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Distance-Regular Antipodal Covering Graphs

R. E. L. ALDRED*

Department of Mathematics, University of Melbourne, Parkville, 3052 Victoria, Australia

AND

C. D. GODSIL

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

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A characterization of distance-regular antipodal coverings of complete bipartite graphs is presented. Using the characterization, particular classes of such graphs and equivalence to well-known combinatorial structures is noted. © 1988 Academic Press, Inc.

INTRODUCTION

In the study of distance-regular graphs there are one or two outstanding open problems which lie at the base of most work done in the area. One of these questions is: When does a given intersection array have a graphical realisation? The main result of this paper was obtained from looking at this very problem with the added constraint that the intersection array be that of an n-fold antipodal covering of a complete bipartite graph.

Before defining the terms used to date we shall state the main result of the paper and then proceed to define the terminology required for the paper.

The chief goal of this paper is to prove the following theorem.

THEOREM. *An n-fold distance-regular antipodal covering of $K_{tn,tn}$ is equivalent to the array $T(h, i, j)$, $1 \leqslant h, j \leqslant n$, $1 \leqslant i \leqslant tn-1$, of subsets of a tn-set, with the following.*

(i) *For each j, $1 \leqslant j \leqslant n$, $T(h, i, j)$ is a resolvable 2-$(tn, t, t-1)$ design.*

* Present address: Southern Illinois University, Carbondale, Illinois 62901, U.S.A.

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(ii) $\sum_{j=1}^{n} |T(h, i, j) \cap T(h', i', j)| = t$, $i \neq i'$.

(iii) *For each quadruple* x, y, j, j', *satisfying* $1 \leqslant x, y \leqslant tn$, $1 \leqslant j, j' \leqslant n$, *and* $x \neq y$, $j \neq j'$, *we have* $x \in T(h, i, j)$ *and* $y \in T(h, i, j')$ *for exactly* t *pairs* h, i.

Furthermore, these are the only antipodal covers of complete bipartite graphs.

In order to do this we must first define the terms of the theorem.

Let G' be a regular graph of diameter d with vertex set VG' and edge set EG'. We say that G' is *distance-regular* if for vertices $u, v \in VG'$ distance i apart, v is adjacent to c_i vertices in VG' distance $i-1$ from u, a_i vertices in VG' distance i from u, and b_i vertices in VG' distance $i+1$ from u and the numbers c_i, a_i, and b_i depend only upon the value of i and not upon the choice of u and v. These numbers are called *intersection numbers*. The values of c_0 and b_d are officially undefined although no inconsistency arises if they are considered to be zero.

If G' is distance-regular there is associated with it an *intersection array*

$$i(G') = \{b_0, b_1, ..., b_{d-1}; c_1, c_2, ..., c_d\}.$$

Since $b_i + a_i + c_i = b_0$ for all $i = 1, 2, ..., d-1$ the intersection array gives us all intersection numbers associated with G'.

There are many things about G' which can be determined from its intersection array. The question of whether or not a given intersection array can have a graphical realization is, as we mentioned earlier, open and quite hard. To date several necessary, or feasibility conditions have been found that an array must satisfy so as not to rule out the possibility of admitting a graphical realization. These conditions do not, however, guarantee the existence of a distance-regular graph with that intersection array. For further theory of distance-regular graphs see [2].

The graph G' is said to be *antipodal* if given any vertices $u, v, w \in VG'$ with v and w distance d from u then either $v = w$ or v is distance d from w. The set of vertices consisting of a vertex and all vertices distanced from it is called an *antipodal block* of G'.

Given a distance-regular antipodal graph of diameter $d \geqslant 3$ there are several observations one can make about the structure of the graph. First, we note that since the diameter is at least three, no two vertices in the same antipodal block have a common neighbour. Distance-regularity demands than an edge between a pair of vertices implies the existence of a perfect matching of the vertices in the corresponding antipodal blocks. This in turn demands that all antipodal blocks in G' have the same number of vertices. With some simple counting arguments we see that no antipodal block can contain more than b_0 vertices.

The observations above suggest an obvious quotient of G' as follows. From G' we form the new graph with $VG = \{\text{antipodal blocks of } G'\}$ and two vertices in VG are adjacent if there is a perfect matching between the vertices in the corresponding antipodal blocks of G'. We say that G' is a *distance-regular antipodal covering* graph of G. Where there should be no confusion caused we shall refer to a such a graph as an *antipodal cover* of G. If each antipodal block of G' contains n vertices then we say that G' is an *n-fold antipodal cover* of G and denote this by $n(G)$.

The theory of antipodal covers has been developed by Gardiner [3] and particular attention has been paid to antipodal covers of distance-regular graphs, it is shown that for a distance-regular graph G with intersection array

$$i(G) = \{b_0, b_1, ..., b_{d-1}; c_1, c_2, ..., c_d),$$

an n-fold antipodal cover G' has intersection array

$$\begin{aligned} i(G') = \{&b_0, b_1, ..., b_{d-1}, c_d(n-1)/n, c_{d-1}, ..., c_1; \\ &c_1, c_2, ..., c_d/n, b_{d-1}, ..., b_1, b_0) \end{aligned} \tag{1}$$

or

$$\begin{aligned} i(G') = \{&b_0, b_1, ..., b_{d-1}, (n-1)\, c_{d+1}, c_d, ..., c_1; \\ &c_1, ..., c_d, c_{d+1}, b_{d-1}, ..., b_1, b_0\}, \end{aligned} \tag{2}$$

where c_{d+1} is a new intersection number to be determined by feasibility conditions.

Restricting our attention to the existence of n-fold antipodal covers of distance-regular graphs gives a great deal more information. In particular instances it enables us to find more restrictive conditions to apply to the intersection array to determine its feasibility.

The main result in this paper is a characterization of the n-fold antipodal covers of complete bipartite graphs.

Main Result

For ease and clarity of notation we first make a few elementary observations about the feasibility of an n-fold antipodal cover of $K_{m,m}$. Since $K_{m,m}$ is bipartite it is clear that $n(K_{m,m})$ must also be bipartite and thus we may assume that $n(K_{m,m})$ has even diameter, that is, has the form shown in (1). So $i(K_{m,m}) = \{m,\ m-1;\ 1,\ m\}$ gives us $i(n(K_{m,m})) = \{m,\ m-1,\ m(n-1)/n,\ 1;\ 1,\ m/n,\ m-1,\ m\}$. The intersection numbers must be integers so we must conclude that $n \backslash m$, i.e., $m = tn$, $t \in \mathbf{Z}^+$. We shall continue then to characterize $n(K_{tn,tn})$.

We wish to determine whether or not the intersection array

$$i(n(K_{tn,tn})) = \{tn, tn-1, t(n-1), 1; i, t, tn-1, tn\} \tag{3}$$

has a graphical realization. The fact that a graphical realization of this array is an n-fold antipodal cover of $K_{tn,tn}$ tells us much about the structure of the putative graph.

We begin the construction by partitioning the vertex set of $n(K_{tn,tn})$ into $2tn$ antipodal blocks of cardinality n, tn blocks $B_1, ..., B_{tn}$ which correspond to fibres over black vertices in $K_{tn,tn}$, and tn blocks which correspond to fibres over white vertices in $K_{tn,tn}$. We shall label the vertices $F_i(j)$, $1 \leqslant j \leqslant n$, $F_1 \in \{B_1, ..., B_{tn}, W_1, ..., W_{tn}\}$.

Now we fix attention on one particular block, say W_{tn}. Every vertex in W_{tn} is adjacent to exactly one vertex in each of the blocks $B_1, ..., B_{tn}$. The antipodality requirement demands that no two vertices in W_{tn} are adjacent to the same $B_i(j)$, $1 \leqslant i \leqslant tn$, $1 \leqslant j \leqslant n$. So we may assume, without loss of generality, that $W_{tn}(j)$ is adjacent to $B_i(j)$ for all $1 \leqslant i \leqslant tn$, $1 \leqslant j \leqslant n$. (That is, the subgraph induced by a block and its neighbourhood is isomorphic to n disjoint copies of $K_{1,tn}$.)

Since $n(K_{tn,tn})$ is bipartite and diameter four, any pair of vertices $W_i(j)$, $W_k(l)$, $1 \leqslant i$, $k \leqslant tn$, $1 \leqslant j$, $l \leqslant n$, $i \neq k$, is distance two apart. From the intersection array (3) we see that any pair of vertices distance two apart has exactly t common neighbours. In particular

(S.1) $W_{tn}(j)$ and $W_k(l)$, $1 \leqslant k \leqslant tn-1$, $1 \leqslant j$, $l \leqslant n$, have exactly t neighbours in common.

Similarly vertices in distinct black blocks are distance two apart and thus have exactly t common neighbours. In particular

(S.2) For each j, $1 \leqslant j \leqslant n$, $B_i(j)$ and $B_{i'}(j)$, $1 \leqslant i$, $i' \leqslant tn$, $i \neq i'$, have exactly t common neighbours one of which is $W_{tn}(j)$.

Consider the following structure:

Let $\Omega = \{1, ..., tn\}$ and $\mathfrak{B}(j) = \{\{k\colon W_i(h)$ is adjacent to $B_k(j)\}$, $1 \leqslant i \leqslant tn-1$, $1 \leqslant h \leqslant n\}$, $1 \leqslant j \leqslant n$. We claim that for each value of j, $1 \leqslant j \leqslant n$, Ω, $\mathfrak{B}(j)$ forms a 2-$(tn, t, t-1)$ design with element set Ω and block set $\mathfrak{B}(j)$.

To prove the claim we return to statements (S.1) and (S.2) above. (S.1) guarantees that each block in $\mathfrak{B}(j)$ contains precisely t elements and (S.2) guarantees that each pair of elements in Ω occurs in exactly $t-1$ blocks of $\mathfrak{B}(j)$. Straightforward counting confirms that all the requirements of a 2-$(tn, t, t-1)$ design are satisfied.

In addition to the design structure we make the following observation. Since $n(K_{tn,tn})$ is antipodal with diameter four we have, adopting the notation $a \sim b$ for a is adjacent to b:

(S.3) $\bigcup_{h=1}^{n} \{k: W_1(h) \sim B_k(j)\} = \{1, ..., tn\}$ and $\{k: W_i(h) \sim B_k(j)\} \cap \{k: W_i(h') \sim B_k(j)\} = \phi$ when $h \neq h'$ for every pair (i, j), $1 \leqslant i \leqslant tn$, $1 \leqslant j \leqslant n$ and $1 \leqslant h, h' \leqslant n$.

From this we deduce that the designs Ω, $\mathfrak{B}(j)$, $1 \leqslant j \leqslant n$, are in fact also resolvable.

The designs defined above arise from requiring that the jth vertices in distinct antipodal blocks share precisely t common neighbours since they are distance two apart and the requirement that every white vertex outside of W_{tn} has exactly t neighbours in common with $W_{tn}(j)$, $1 \leqslant j \leqslant n$ because they are distance two apart.

We must now ensure that all other pairs of vertices distance two apart, namely arbitrary pairs of black vertices from distinct blocks and arbitrary pairs of white vertices from distinct blocks, share t common neighbours. To do this we define a new structure which, for the want of a better name, we shall call a *tank-trap*. A tank-trap is an $n \times tn - 1 \times n$ array the entries of which are defined by

$T(h, i, j) = \{k: W_i(h) \sim B_k(j)\} =$ the set of blocks in $\mathfrak{B}(j)$ determined by $W_i(h)$ and $W_{tn}(j)$ having t common neighbours.

(So $T(h, i, j)$ describes the set of black blocks which have the hth vertex of the ith white block adjacent to their jth vertex.)

We see then that by holding j fixed, $T(h, i, j)$, $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant tn - 1$, gives us the resolvable design described above arranged in parallel classes. That is, each column $T(h, i, j)$, $1 \leqslant h \leqslant n$, is a parallel class of the design fixed by j. To make sure that pairs of white vertices from distinct blocks other than W_{tn} share exactly t common neighbours we require that

$$\sum_{j=1}^{n} |T(h, i, j) \cap T(h', i', j)| = t \qquad \text{for} \quad i \neq i'.$$

Ensuring that each pair of black vertices from distinct blocks (other than both jth vertices) shares exactly t common neighbours requires that:

For each quadruple x, y, j, j', satisfying $1 \leqslant x$, $y \leqslant tn$, $1 \leqslant j, j' \leqslant n$, and $x \neq y$, $j \neq j'$, we have $x \in T(h, i, j)$ and $y \in T(h, i, j')$ for exactly t pairs h, i.

Having achieved the correct neighbourhood relationships for all possible pairs of distance two vertices and having maintained the antipodality and bipartiteness throughout means that the entire graph is determined by the tank-trap above.

In summary we conclude from the construction above that:

THEOREM. *$n(K_{tn,tn})$ is equivalent to the array $T(h, i, j)$, $1 \leqslant h, j \leqslant n$, $1 \leqslant i \leqslant tn-1$, of subsets of a tn-set with the following.*

(i) *For each j, $1 \leqslant j \leqslant n$, $T(h, i, j)$ is a resolvable 2-$(tn, t, t-1)$ design.*

(ii) $\sum_{j=1}^{n} |T(h, i, j) \cap T(h', i', j)| = t$, $i \neq i'$.

(iii) *For each quadruple x, y, j, j', satisfying $1 \leqslant x, y \leqslant tn$, $1 \leqslant j, j' \leqslant n$, and $x = y$, $j \neq j'$, we have $x \in T(h, i, j)$ and $y \in T(h, i, j')$ for exactly t pairs h, i.*

Furthermore, these are the only antipodal covers of complete bipartite graphs.

While the theorem is deduced from the position of having the graph and forming the tank-trap from that, it is clear that, given the comment in parentheses following the definition of tank-trap, we can construct an antipodal covering graph from a tank-trap.

EXAMPLES. The maximal coverings $n(K_{n,n})$ require that we vacuously satisfy the design requirement of the tank-trap as $t = 1$. The "design" is then $n-1$ listings of $(1, ..., n)$ as singletons. We are completely at liberty to arrange the entries so that $T(h, i, 1) = h$, $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant n-1$.

Now form $T'(h, i, j)$, $1 \leqslant h, j \leqslant n$, $1 \leqslant i \leqslant n$, the canonical extension of $T(h, i, j)$, $1 \leqslant h, j \leqslant n$, $1 \leqslant i \leqslant n-1$, by taking $T'(h, i, j) = (T(h, i, j)$ in their mutual range of indices and $T'(h, n, j) = h$, $1 \leqslant h, j \leqslant n$.

For each value of j, $2 \leqslant j \leqslant n$, $T'(h, i, j)$, $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant n$, is a Latin Square. Furthermore, the Latin Squares associated with distinct values of j are orthogonal. The equivalence demonstrated here between the tank-trap and a set of $n-1$ mutually orthogonal Latin Squares is the same as showing that $n(K_{n,n})$ is equivalent to a projective plane of order n with a distinguished flag, as was shown by Gardiner [4].

Consider the case where $n = 2$. The tank-trap is then completely determined by the 2-$(2t, t, t-1)$ resolvable design as the second design is the same as the first (with the blocks in each parallel class interchanged). The requirement on relationships between designs becomes a requirement on the one design to be affine resolvable (with block intersection parameter $t/2$ which clearly demands that t is even).

Consider the design $T(h, i, 1)$, $1 \leqslant h \leqslant 2$, $1 \leqslant i \leqslant 2t-1$. Arrange the design so that $T(1, i, 1)$, $T(2, i, 1)$, $1 \leqslant i \leqslant 2t-1$, is a parallel class and so that for all i in this range we have $1 \in T(1, i, 1)$. We now produce a matrix through a canonical procedure as follows:

The first row of the matrix consists of $2t$ ones. For each subsequent row

$$A(i, j) = \begin{cases} 1 & \text{if } j \in T(1, i, 1), \\ -1 & \text{otherwise.} \end{cases}$$

Call the resulting matrix M. The design parameters ensure that each column differs from every other column in precisely t places. Similarly the rows are orthogonal. Consequently $MM^T = 2tI$ and so M is Hadamard.

The equivalence between 2-fold antipodal covers of complete bipartite graphs and Hadamard matrices has been established separately [1] but we can immediately deduce the equivalence of Hadamard matrices and affine resolvable 2-$(2t, t, t-1)$ designs from the above discussions.

Using affine resolvability it is possible to construct tank-traps reasonably easily and hence establish the existence of (indeed give the plans for the construction of) antipodal covers of certain complete bipartite graphs.

An affine plane of order n has associated with it an affine resolvable 2-$(n^2, n, 1)$ design such that blocks from distinct parallel classes have exactly one element in common. Using $n-1$ copies of the design associated with an affine plane of order n we can construct a tank-trap $T(h, i, j)$, $1 \leqslant h, j \leqslant n$, $1 \leqslant i \leqslant n^2 - 1$, as follows:

Take one copy of the design and arrange it into an array with parallel classes as rows. We want to consider each row as a cyclic ordering of the numbers from 1 to n. If n is a prime we can find $n-1$ such orderings so that no pair of orderings has the same ordered pair of elements the same (left to right) distance apart. That is, the distance from a to b moving left to right is different in every ordering.

Labelling the columns of the array obtained from the first design we associate with it one of our $n-1$ cyclic orderings. We associate a distinct cyclic ordering with each of the copies of the design and extend the array by adding the remaining designs underneath the first except that the columns are permuted according to the cyclic ordering associated with the particular design. This gives us the $T(h, i, 1)$, $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant n^2 - 1$, "face" of the tank-trap. The remaining "faces" $T(h, i, j)$, $1 \leqslant h, j \leqslant n$, $1 \leqslant i \leqslant n^2 - 1$, are obtained from the first by cyclically permuting the columns of the first face ($T(h, i, 1)$, $1 \leqslant i \leqslant n^2 - 1$, h fixed is a column of the first face) $j-1$ notches to the right.

It is easy to check that the resulting structure is a tank-trap and hence that there exist n-fold antipodal covers of K_{n^2, n^2} for all n prime. It is conjectured at this time that the above procedure can only be applied when n is prime as there are no other values of n for which one can find the $n-1$ cyclic orderings required.

There are other known constructions for antipodal covers of complete bipartite graphs including a construction of $n(K_{n^k, n^k})$ for n a prime power due to J. Shawe-Taylor and one due to R. Mathon for $n(K^{k;k}_{2n,2n})$, where n is a prime power. The first uses vector spaces over Galois fields and the second uses orthogonal arrays. Ivanov, Ivanov, and Faradzev [6] have established the equivalence of n-fold covers of complete bipartite graphs to matrices with certain properties. (Note: The Mathon and Shawe-Taylor constructions are unpublished and were communicated in [5], [7].)

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